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PHASE TRANSITIONS FOR RANDOM GEOMETRIC PREFERENTIAL ATTACHMENT GRAPHS

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Abstract

Vertices arrive sequentially in space and are joined to existing vertices at random according to a preferential rule combining degree and spatial proximity. We investigate phase transitions in the resulting graph as the relative strengths of these two components of the attachment rule are varied.

Previous work of one of the authors showed that when the geometric component is weak, the limiting degree sequence mimics the standard Barabási–Albert preferential attachment model. We show that at the other extreme, in the case of a sufficiently strong geometric component, the limiting degree sequence mimics a purely geometric model, the on-line nearest-neighbour graph, for which we prove some extensions of known results. We also show the presence of an intermediate regime, with behaviour distinct from both the on-line nearest-neighbour graph and the Barabási–Albert model; in this regime, we obtain a stretched exponential upper bound on the degree sequence.

Keywords: Random spatial network; preferential attachment; on-line nearest-neighbour graph; degree sequence

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1. Introduction

Stochastic models in which a network evolves via the sequential addition of new nodes, each connected by an edge to an existing node in the graph according to some

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probabilistic rule, have been the subject of an explosion of interest over the past decade or so, motivated by real-world graphs such as those associated with social networks or the internet. The subject of this paper is a model whose connectivity rule combines degree-based preferential attachment with a spatial component; we describe our model in detail below. This model, previously studied in [11], is a variant of the geometric preferential attachment model of Flaxman *et al.* [7,8], which itself can be viewed as a generalization of an earlier model of Manna and Sen [12]. A continuous time model with a similar flavour has recently been studied by Jacob and Mörters [9].

In a sense to be explained in this paper, the behaviour of the model studied here interpolates between pure preferential attachment (essentially the well-known Barabási–Albert model) and a purely geometric model (the on-line nearest-neighbour graph). It was shown in [11] that for a sufficiently weak geometric component of the attachment rule, the limiting degree distribution coincides with that of the Barabási–Albert model, which famously has a 'scale-free' or 'power-law' degree distribution [2,10].

The focus of the present paper is the complementary setting, in which the geometric component has a significant impact. We show that in the extreme case of a dominant geometric effect, the model behaves similarly to the on-line nearest-neighbour graph, which by contrast has a degree distribution with exponential tails (cf [1]). We also study an intermediate regime in which the model behaves differently from both of the extreme cases, and in which the degree distribution satisfies a stretched exponential tail bound. Thus we demonstrate the existence of non-trivial phase transitions.

In the next section we describe our models precisely and state our main results.

2. Random spatial graph models and main results

2.1. Notation

Write $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, and $\mathbb{R}_+ := [0, \infty)$. The vertices of our graphs are associated with *sites* in $S \subset \mathbb{R}^d$, assumed to be compact, convex, and of positive d-dimensional Lebesgue measure. The locations of the sites are $X_0, X_1, \ldots$, independent random variables with density f supported on S. For $n \in \mathbb{N}$, set $\mathcal{X}_n :=$

$\{X_0, \ldots, X_n\}$. We assume throughout that f is bounded away from 0 and ∞:

$$0 < \lambda_0 := \inf_{x \in S} f(x) \leq \sup_{x \in S} f(x) =: \lambda_1 < \infty. \tag{2.1}$$

We write $\| \cdot \|$ for the Euclidean norm on $\mathbb{R}^d$, and $\rho(x, y) = \|x - y\|$ for the Euclidean distance between x and y in $\mathbb{R}^d$. Denote by $B(x; r)$ the open Euclidean d-ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$. Throughout we understand $\log x$ to stand for $\max\{0, \log x\}$. Let $\#A$ denote the number of elements of a finite set A.

2.2. On-line nearest-neighbour graph

The on-line nearest-neighbour graph (ONG) is constructed on points arriving sequentially in $\mathbb{R}^d$ by connecting each point after the first to its nearest predecessor. The ONG is a natural and basic model of evolving spatial networks. It is a special case (or limiting case) of several models that have appeared in the literature, including a version of the 'FKP' network model [1, 5] and geometric preferential attachment models such as [7, 11, 12] (specifically, it is the '$\alpha = -\infty$' case of the model of Manna and Sen [12]); one contribution of the present paper is to explore this latter connection. The name 'on-line nearest-neighbour graph' was apparently introduced by Penrose [16].

In the ONG on $(X_0, \ldots, X_n)$, the nth edge ($n \in \mathbb{N}$) is between X_n and its (a.s. unique) nearest neighbour among $\mathcal{X}_{n-1}$. In other words, writing

$$\eta_1(n) := \underset{i \in \{0, \ldots, n-1\}}{\arg\min} \ \rho(X_n, X_i), \tag{2.2}$$

the ONG on $(X_0, \ldots, X_n)$ consists of the edges $(i, \eta_1(i))$ for $1 \leq i \leq n$; it is natural to view these as directed edges when constructing the graph, but we largely treat them as undirected. We call $X_{\eta_1(n)}$ the *on-line nearest neighbour* of X_n.

Let $\deg_n(i)$ denote the degree of vertex i in the ONG on $(X_0, \ldots, X_n)$. Let $N_n^{\text{ONG}}(k)$ denote the number of vertices with degree at least k in the ONG on $(X_0, \ldots, X_n)$:

$$N_n^{\text{ONG}}(k) = \sum_{i=0}^{n} \mathbf{1}\{\deg_n(i) \geq k\}.$$

We study the *asymptotic degree sequence*, i.e., the asymptotic proportion of vertices with degree at least k (for each k); for convenience we work with $n^{-1} N_n^{\text{ONG}}(k)$.

Part of the statement of our main result on the ONG, Theorem 2.1 below, is that $\lim_{n \to \infty} n^{-1} \mathbb{E}[N_n^{\text{ONG}}(k)]$ *exists* for each k; this was stated, apparently without proof, in [1, §2], but can be justified for the ONG using stabilization arguments of Penrose [16], as we explain in Section 6 below. Stabilization also gives an explicit description of the limit in terms of a version of the ONG defined on an infinite Poisson point process, as we describe next; in particular, the limit depends only on d and not on S or f.

Let $\mathcal{H}$ be a unit-rate homogeneous Poisson point process on $\mathbb{R}^d \times [0,1]$; the $[0,1]$-valued *marks* play the role of time in the finite ONG. For $u \in [0,1]$, let $\mathcal{H}_u := \mathcal{H} \cap (\mathbb{R}^d \times [0,u])$, those Poisson points with marks in $[0,u]$. For $x, y \in \mathbb{R}^d$ let $B_x(y)$ be the open ball with centre y whose boundary includes x. Given $x \in \mathbb{R}^d$ and $u \in [0,1]$, let

$$\xi(x,u;\mathcal{H}) := 1 + \sum_{(y,v)\in\mathcal{H},\, v>u} \mathbf{1}\{\mathcal{H}_v \cap (B_x(y) \times [0,1]) = \{(y,v)\}\}.$$

By stabilization for the ONG (see [16]), $\xi(x,u;\mathcal{H}) < \infty$ a.s. for any $x \in \mathbb{R}^d$ and any $u \in (0,1)$. We call $\xi(x,u;\mathcal{H})$ the *degree* of (x,u) in the *infinite Poisson on-line nearest-neighbour graph*, defined locally by joining each point to the nearest Poisson point with mark equal to or less than the mark of the given point; note that (x,u) itself need not be in $\mathcal{H}$. Let U denote a uniform $[0,1]$ random variable, independent of $\mathcal{H}$.

Theorem 2.1. *Let $d \in \mathbb{N}$. Then for any $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} n^{-1} N_n^{\mathrm{ONG}}(k) = \lim_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] = \mathbb{P}[\xi(0,U;\mathcal{H}) \geq k] =: \rho_k, \tag{2.3}$$

the first limit equality holding a.s. and in L^1. Here $\rho_k \in [0,1]$ are nonincreasing with $\rho_1 = 1$, $\lim_{k\to\infty}\rho_k = 0$, and $\sum_{k\in\mathbb{N}}\rho_k = 2$. Moreover, there exist finite positive constants A, A', C, C' such that, for all $k \in \mathbb{N}$,

$$A' \mathrm{e}^{-C'k} \leq \rho_k \leq A\mathrm{e}^{-Ck}, \tag{2.4}$$

and, more precisely,

$$\frac{1}{2}\log\left(1 + (2^{2d}-1)^{-1}\right) \leq \liminf_{k\to\infty}\left(-k^{-1}\log\rho_k\right) \leq \limsup_{k\to\infty}\left(-k^{-1}\log\rho_k\right) \leq 1. \tag{2.5}$$

Finally, there exists a constant $D < \infty$ for which, a.s., for all n sufficiently large,

$$\max_{0\leq i\leq n} \deg_n(i) \leq D\log n. \tag{2.6}$$

This result extends a result of Berger *et al.* [1]. Specifically, [1, Theorem 3] showed

$$A'\mathrm{e}^{-C'k} \leq \liminf_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \leq \limsup_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \leq A\mathrm{e}^{-Ck},$$

degree	1	2	3	4	5	6	7	8	9	10
$d = 1$	0.4728	0.2675	0.1394	0.0670	0.0304	0.0132	0.0056	0.0024	0.0001	0.0000
$d = 2$	0.4777	0.2636	0.1369	0.0668	0.0308	0.0137	0.0060	0.0026	0.0001	0.0000
$d = 100$	0.4999	0.2501	0.1250	0.0625	0.0312	0.0156	0.0078	0.0039	0.0002	0.0001

TABLE 1: Estimated $\mathbb{P}[\xi(0, U; \mathcal{H}) = k]$ (4dp) for $1 \leq k \leq 10$, for $d \in \{1, 2, 100\}$. For each d, the estimates are based on 500 simulations with $n = 10^5$ for f uniform on the d-dimensional torus, to avoid finite-sample boundary effects.

in the special case where $d = 2$ and f is the indicator of the unit square $S = (0, 1)^2$. Our proof of Theorem 2.1, which we give in Section 6 below, is based in part on the proof of the analogous result in [1], with additional arguments required to obtain the existence of the limit and the almost-sure convergence in (2.3). Some extra work is also needed to obtain the quantitative bounds in (2.5): the $d = 2$ case of the lower bound, $\frac{1}{2} \log \frac{16}{15}$, is contained in the argument of [1]; the other bounds are new.

Remark 2.1. In view of (2.5), it is natural to conjecture that, for each $d \in \mathbb{N}$,

$$\lim_{k \to \infty} \left(-k^{-1} \log \rho_k \right) = \mu(d) \text{ exists in } (0, 1];$$

the upper bound of 1 comes from (2.5). In [18, §7.6.5] it was conjectured that one might have $\mu(d) = \mu = 1$. The simpler, non-spatial, *uniform attachment* model in which vertex n is connected uniformly at random to a vertex from $\{0, 1, \dots, n-1\}$ leads to an analogous result with $\mu = \log 2$, as follows from [2, §4]. We think it unlikely that $\mu(d) \in \{1, \log 2\}$ for any $d \in \mathbb{N}$; we conjecture, however, that $\lim_{d \to \infty} \mu(d) = \log 2$. Simulations suggest that $\mu(1) \approx 0.79$, $\mu(2) \approx 0.77$, and $\mu(100) \approx 0.69$: cf Table 1.

2.3. Geometric preferential attachment graph

The geometric preferential attachment (GPA) model that we study is as follows; often our notation coincides with [11]. We define a (random) sequence of finite graphs $G_n = (V_n, E_n)$, $n \in \mathbb{N}$. The vertex set of G_n is $V_n = \{0, 1, \dots, n\}$. For $v \in V_n$, we denote by $\deg_n(v)$ the degree of v in the GPA graph G_n (viewed as an undirected graph); this notation is the same as for degrees in the ONG, but the graph under consideration will be clear in context.

The construction uses an *attractiveness function* $F : (0, \infty) \to (0, \infty)$. Recall that $X_0, X_1, \dots$ are random sites in S. For simplicity, we start with $G_1 = (V_1, E_1)$ consisting of vertices with labels 0 and 1 joined by a single edge, so $V_1 = \{0, 1\}$ and $E_1 = \{(1, 0)\}$. Vertices 0 and 1 are associated with sites X_0 and X_1 in S, respectively.

We proceed via iterated addition of vertices to construct $G_{n+1} = (V_{n+1}, E_{n+1})$ from $G_n = (V_n, E_n)$, $n \in \mathbb{N}$. Given G_n, $n \in \mathbb{N}$, and the spatial locations $\mathcal{X}_n$ of its vertices, we add a vertex with label $n+1$ at site $X_{n+1} \in S$, and we add a new edge $(n+1, v_{n+1})$ where v_{n+1} is chosen randomly from V_n with distribution specified by

$$\mathbb{P}[v_{n+1} = v \mid G_n, \mathcal{X}_{n+1}] = \frac{\deg_n(v) F(\rho(X_v, X_{n+1}))}{D_n(X_{n+1})}, \quad v \in V_n, \tag{2.7}$$

where for $n \in \mathbb{N}$ and $x \in S$, $D_n(x) := \sum_{v \in V_n} \deg_n(v) F(\rho(X_v, x))$.

We call G_n so constructed a GPA graph with attractiveness function F. In [11], it was assumed that $\int_S F(\rho(x, y)) f(y) \mathrm{d}y < \infty$, so F cannot blow up too rapidly at 0. In this paper, our primary interest is in F for which this condition is not satisfied.

2.4. Strong geometric regime

For $\gamma > 1$, define F_γ for $r > 0$ by $F_\gamma(r) := \exp\{(\log(1/r))^\gamma\}$. Note that $F_\gamma(r)$ blows up at 0 faster than r^{-s} for any power s. Recall that the convention $\log x \equiv \max\{0, \log x\}$ is in force, so $F_\gamma(r) = 1$ for $r \geq 1$. Also, $F_\gamma(r)$ is strictly decreasing for $r \in (0, 1)$, with $F_\gamma(r) \to \infty$ as $r \downarrow 0$.

Our main result in this setting (i) gives an almost-sure degree bound analogous to (2.6) above for the ONG, and (ii) shows that the limiting degree sequence for the GPA graph is the same as for the ONG, for a strong enough geometric component to the interaction (under the condition $\gamma > 3/2$). Let $N_n^{\mathrm{GPA}}(k)$ denote the number of vertices with degree at least k in the GPA graph G_n.

Theorem 2.2. *Suppose that $F = F_\gamma$ for some $\gamma > 1$.*

(i) For any $\nu \in (0, 1)$ with $\nu > 2 - \gamma$, a.s., for all n sufficiently large,

$$\max_{0 \leq i \leq n} \deg_n(i) \leq \exp\{(\log n)^\nu\}. \tag{2.8}$$

(ii) Suppose that $\gamma > 3/2$. Then $\lim_{n\to\infty} \mathbb{P}[v_n = \eta_1(n)] = 1$ and

$$\lim_{n\to\infty} n^{-1} \mathbb{E} \sum_{i=1}^{n} \mathbf{1}\{v_i \neq \eta_1(i)\} = 0. \tag{2.9}$$

Moreover, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{-1} N_n^{\mathrm{GPA}}(k) = \lim_{n\to\infty} n^{-1} \mathbb{E}[N_n^{\mathrm{GPA}}(k)] = \rho_k, \tag{2.10}$$

the first limit equality holding in L^1, where ρ_k is as in Theorem 2.1.

We give the proof of Theorem 2.2 in Section 4.

Conjecture 2.1. *We suspect the conclusion of Theorem 2.2(ii) to hold for* any $\gamma > 1$.

2.5. Intermediate regime: power-law attractiveness

Take $F(r) = r^{-s}$ for $s \in (0, \infty)$. The next result contrasts with (2.9) in the strong geometric attraction regime, and shows that in this case, in expectation, there is a non-negligible proportion of vertices not connecting to their nearest neighbour.

Theorem 2.3. *Suppose that $F(r) = r^{-s}$ for $s \in (0, \infty)$. Then* $\limsup_{n\to\infty} \mathbb{P}[v_n = \eta_1(n)] < 1$ *and*

$$\liminf_{n\to\infty} n^{-1}\mathbb{E}\sum_{i=1}^{n} \mathbf{1}\{v_i \neq \eta_1(i)\} > 0. \tag{2.11}$$

It was proved in Theorem 2.1 of [11] that in the case $s \in (0, d)$, under certain conditions on S and f, the degree distribution of the GPA graph converges to a power-law distribution, as in the Barabási–Albert model: $\lim_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{GPA}}(k)] = r_k$ where $r_k \sim 2k^{-2}$ as $k \to \infty$. In contrast, the next result gives a stretched exponential upper bound for the tail of the degree distribution when $s > d$.

Theorem 2.4. *Suppose that $F(r) = r^{-s}$ for $s > d$. For any $\gamma \in (0, \frac{s-d}{2s-d})$, there exists a constant $C < \infty$ such that, for all k,*

$$\limsup_{n\to\infty} n^{-1}N_n^{\mathrm{GPA}}(k) \leq C\mathrm{e}^{-k^{\gamma}}, \ \textit{a.s., and} \ \limsup_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{GPA}}(k)] \leq C\mathrm{e}^{-k^{\gamma}}.$$

This result confirms the presence of a phase transition in the character of the degree distribution at $s = d$, as intimated in [11, §5] and in line with $d \in \{1, 2\}$ simulation results of [12] (who actually conjectured that the phase transition point was $s = d-1$) and [6] (who did suggest $s = d$ for the transition). The stretched exponential for $s > d$ is also consistent with the observations of [6,12]. We remark that as $s \to \infty$, Theorem 2.4 gives an upper bound of order almost $\mathrm{e}^{-\sqrt{k}}$.

The rest of the paper is organized as follows. In Section 3 we collect some preparatory lemmas. Section 4 deals with the strong geometric regime, culminating in the proof of Theorem 2.2. Section 5 deals with the case of power-law attractiveness, and presents the proofs of Theorems 2.3 and 2.4. Finally, Section 6 deals with the ONG and gives the proof of Theorem 2.1.

3. Preliminaries to the proofs

First we state a basic property of the set S, under our standing assumptions. Let ω_d be the volume of the unit-radius Euclidean d-ball, and set $\mathrm{diam}(S) := \sup_{x,y \in S} \rho(x,y)$.

Lemma 3.1. *There exists $\delta_S > 0$ such that, for all $r \in [0, \mathrm{diam}(S)]$,*

$$\inf_{x \in S} |B(x;r) \cap S| \geq \delta_S \omega_d r^d.$$

Proof. Since S is convex, compact, and of positive measure, there exist $x_0 \in S$ and $r_0 > 0$ such that $B(x_0; r_0)$ is contained in the interior of S. It suffices to suppose that either (i) $\rho(x, x_0) \geq 2r_0$, or (ii) $\rho(x, x_0) \leq r_0/2$. To see this, suppose that $r_0/2 < \rho(x, x_0) < 2r_0$. Then we may carry out the argument for case (i) after having replaced r_0 by $r_0/4$, introducing only a constant multiplicative factor into the argument.

So now suppose that (i) holds. For $r \leq r_0$, let $C(x,r)$ denote the cone with apex x, axis passing through x_0, and half-angle $\theta(x,r) = \sin^{-1}(r/\rho(x,x_0))$. Since $\rho(x,x_0) \leq \mathrm{diam}(S)$, $\theta(x,r) \geq \theta(r) := \sin^{-1}(r/\mathrm{diam}(S))$. By construction and convexity of S, $C(x,r) \cap S$ contains the cone segment $\{y \in C(x,r) : \rho(x,y) \leq \rho(x,x_0) \cos\theta(x,r)\}$. So, if $\rho(x,x_0) \geq 2r_0$, then $B(x;r) \cap S$ contains the cone segment $\{y \in C(x,r) : \rho(x,y) \leq r \wedge r_0\}$, which has volume bounded below by $c_d \theta(r)^{d-1} r$, provided $r \leq r_0$, where $c_d > 0$ is an absolute constant. Hence $|B(x;r) \cap S|$ is bounded below by a constant times r^d, for all $r \leq r_0$. The same conclusion follows if $r \in (r_0, \mathrm{diam}(S))$, using the lower bound $c_d \theta(r_0)^{d-1} r_0 \geq c'_d (r_0/\mathrm{diam}(S))^d r^d$ for $c'_d > 0$ not depending on r.

Finally, in case (ii), we have that $B(x;r) \cap S$ contains the ball $B(x; r \wedge (r_0/2))$, and a similar argument to that for part (i) completes the proof.

We next give some basic results on nearest-neighbour distances. For $n \in \mathbb{N}$, let

$$Z_n := \rho(X_n; \mathcal{X}_{n-1}) := \min_{0 \leq i \leq n-1} \rho(X_n, X_i) = \rho(X_n, X_{\eta_1(n)}),$$

the distance from X_n to its on-line nearest neighbour. Write $x^+ := x \mathbf{1}\{x > 0\}$.

Lemma 3.2. *Let $\delta_S > 0$ be the constant in Lemma 3.1. Then for $r > 0$,*

$$\mathbb{P}[Z_n \geq r] \leq (1 - \delta_S \lambda_0 \omega_d r^d)^n \mathbf{1}\{r \leq \mathrm{diam}(S)\}; \tag{3.1}$$

$$\mathbb{P}[Z_n \geq r] \geq ((1 - \lambda_1 \omega_d r^d)^+)^n \mathbf{1}\{r \leq \mathrm{diam}(S)\}. \tag{3.2}$$

Proof. Conditional on X_n, we have, for any $r > 0$, a.s.,

$$\mathbb{P}[Z_n \geq r \mid X_n] = \mathbb{P}[S \cap B(X_n; r) \cap \mathcal{X}_{n-1} = \emptyset \mid X_n]$$
$$= \left(1 - \int_{S \cap B(X_n;r)} f(x)\mathrm{d}x\right)^n. \tag{3.3}$$

Note that $\mathbb{P}[Z_n > \operatorname{diam}(S)] = 0$, so it suffices to suppose that $r \leq \operatorname{diam}(S)$. Then,

$$\delta_S \omega_d r^d \leq |S \cap B(X_n; r)| \leq \omega_d r^d, \text{ a.s.,} \tag{3.4}$$

for all $r \leq \operatorname{diam}(S)$, by Lemma 3.1. It follows from (3.3) and (2.1) that $\mathbb{P}[Z_n \geq r \mid X_n] \leq (1 - \lambda_0 |S \cap B(X_n; r)|)^n$, which, with the first inequality in (3.4), gives (3.1). Similarly, (3.2) follows from (3.3) and (2.1) with the second inequality in (3.4).

Next we state a simple but useful result on degrees, which is basically Markov's inequality. Write $N_n(k)$ for $N_n^{\mathrm{GPA}}(k)$ or $N_n^{\mathrm{ONG}}(k)$. Either graph is a tree, so using the degree sum formula and interchanging the order of summation we obtain

$$2n = \sum_{i=0}^{n} \deg_n(i) = \sum_{i=0}^{n} \sum_{k \geq 1} \mathbf{1}\{\deg_n(i) \geq k\} = \sum_{k \geq 1} N_n(k).$$

So for any $k_0 \in \mathbb{N}$, $2n \geq \sum_{k=1}^{k_0} N_n(k) \geq k_0 N_n(k_0)$, since $N_n(\,\cdot\,)$ is nonincreasing. Thus

$$N_n(k) \leq 2n/k, \text{ a.s., for any } k \text{ and } n. \tag{3.5}$$

Finally, we introduce some notation for dealing with conditional probabilities. Let $\mathcal{F}_n := \sigma(\mathcal{X}_n, v_2, v_3, \ldots, v_{n-1})$, the σ-algebra generated by the sites up to and including X_n and by the edge choices made on previous steps. Then $\mathcal{F}_n$ contains all the information about G_{n-1} as well as $X_0, \ldots, X_n$, and (2.7) can be expressed as

$$\mathbb{P}[v_n = v \mid \mathcal{F}_n] = \frac{\deg_{n-1}(v) F(\rho(X_v, X_n))}{D_{n-1}(X_n)}, \quad v \in \{0, \ldots, n-1\}. \tag{3.6}$$

Also set $\tilde{\mathcal{F}}_n := \sigma(\mathcal{X}_n, v_2, v_3, \ldots, v_{n-1}, v_n)$, which adds to $\mathcal{F}_n$ information about G_n.

4. Proofs for strong geometric regime

In this section we prove our results from Section 2.4. First we outline the central idea of the proof of Theorem 2.2, to show that X_n joins to its on-line nearest neighbour $X_{\eta_1(n)}$ with probability $1 - o(1)$ (cf Lemma 4.3). By (3.6), this probability satisfies

$$\mathbb{P}[v_n = \eta_1(n) \mid \mathcal{F}_n] = \frac{\deg_{n-1}(\eta_1(n)) F(Z_n)}{D_{n-1}(X_n)}.$$

For $F = F_\gamma$, the fact that F_γ is decreasing and the degree sum formula give

$$D_{n-1}(X_n) = \sum_{i=0}^{n-1} \deg_{n-1}(i) F_\gamma(\rho(X_i, X_n)) \leq 2n F_\gamma(W_n) + \deg_{n-1}(\eta_1(n)) F_\gamma(Z_n),$$

where W_n is the distance from X_n to its *second* nearest neighbour among $\mathcal{X}_{n-1}$, so

$$\mathbb{P}[v_n = \eta_1(n) \mid \mathcal{F}_n] \geq 1 - \frac{2nF_\gamma(W_n)}{F_\gamma(Z_n)}.$$

With probability $1 - o(1)$, $W_n > Z_n + \theta_n$ where $\theta_n = o(n^{-1/d})$, so to show $\mathbb{P}[v_n = \eta_1(n)] = 1 - o(1)$ it suffices to show that, as $n \to \infty$,

$$\frac{nF_\gamma(Z_n + \theta_n)}{F_\gamma(Z_n)} \to 0, \text{ in probability.}$$

Taylor's formula shows that this holds provided $\gamma > 2$. Further progress requires control of both the vertex degrees and the number of 'plausible alternatives' for v_n.

Theorem 2.2(i) gives sufficient control of degrees to achieve the $\gamma > 3/2$ case of Theorem 2.2(ii). To achieve $\gamma > 1$ (cf Conjecture 2.1) seems to need tighter control, and a technique that enables one to replace almost-sure upper bounds growing with n by 'typical' statistics, as might be available given some suitable stabilization property, such as that enjoyed by ONG. This seems to be a challenging problem.

For $\nu \in (0,1)$ and $n \geq 2$ set $\beta(n,\nu) := n^{-1/d} \exp\{(\log n)^\nu\}$, and let

$$E(n,\nu) := \{\rho(X_{v_n}, X_n) \geq \beta(n,\nu)\},$$

the event that the edge from vertex n connects to any vertex outside $B(X_n; \beta(n,\nu))$.

Lemma 4.1. *Suppose that $F = F_\gamma$ for some $\gamma > 1$ and that $\nu \in (0,1)$ with $\nu > 2 - \gamma$. Then for any $p < \infty$, as $n \to \infty$,*

$$\mathbb{P}[E(n,\nu)] = O(\exp\{-\gamma d^{1-\gamma}(1 + o(1))(\log n)^{\gamma+\nu-1}\}) = O(n^{-p}).$$

Proof. Note that for any $\nu \in (0,1)$,

$$F_\gamma(\beta(n,\nu)) = \exp\left\{d^{-\gamma}(\log n)^\gamma - \gamma d^{1-\gamma}(1 + o(1))(\log n)^{\gamma+\nu-1}\right\}. \tag{4.1}$$

We obtain from (3.1) and (2.1) that

$$\mathbb{P}[Z_n > \beta(n,\nu)] = O(\exp\{-\delta_S \lambda_0 \omega_d \exp\{d(\log n)^\nu\}\}) = O(\exp\{-(\log n)^K\}), \tag{4.2}$$

for any $K < \infty$. Fix $\nu \in (0,1)$ and choose $\nu' \in (0,\nu)$. Then

$$\mathbb{P}[E(n,\nu)] \leq \mathbb{P}[Z_n > \beta(n,\nu')] + \mathbb{P}[E(n,\nu) \mid Z_n \leq \beta(n,\nu')]. \tag{4.3}$$

Suppose that $Z_n \leq \beta(n,\nu')$. Then, since $F_\gamma(r)$ is nonincreasing in $r > 0$,

$$\deg_{n-1}(\eta_1(n)) F_\gamma(\rho(X_{\eta_1(n)}, X_n)) \geq F_\gamma(Z_n) \geq F_\gamma(\beta(n,\nu')),$$

so that $D_{n-1}(X_n) \geq F_\gamma(\beta(n,\nu'))$, given $Z_n \leq \beta(n,\nu')$. On the other hand, any vertex $j < n$ with $X_j \notin B(X_n; \beta(n,\nu))$ has $\deg_{n-1}(j) F_\gamma(\rho(X_j, X_n)) \leq n F_\gamma(\beta(n,\nu))$, using the crude bound $\deg_{n-1}(j) \leq n$. Hence, by (2.7) and (4.1),

$$\begin{aligned}
\mathbb{P}[E(n,\nu) \mid Z_n \leq \beta(n,\nu')] &= \sum_{j=0}^{n-1} \mathbb{P}[\{v_n = j\} \cap E(n,\nu) \mid Z_n \leq \beta(n,\nu')] \\
&\leq \frac{n^2 F_\gamma(\beta(n,\nu))}{F_\gamma(\beta(n,\nu'))} \\
&= O\left(\exp\left\{2\log n - \gamma d^{1-\gamma}(1+o(1)) \left((\log n)^{\gamma+\nu-1} - (\log n)^{\gamma+\nu'-1}\right)\right\}\right) \\
&= O\left(\exp\left\{-\gamma d^{1-\gamma}(1+o(1))(\log n)^{\gamma+\nu-1}\right\}\right),
\end{aligned} \tag{4.4}$$

provided that $\gamma + \nu - 1 > 1$, i.e., $\nu > 2 - \gamma$, which we can ensure by choosing $\nu \in (0,1)$ close enough to 1 since $\gamma > 1$. The result now follows from (4.2), (4.3) and (4.4).

The next result is a bound on degrees that amounts to Theorem 2.2(i), and which will also be an ingredient in our proof of Theorem 2.2(ii).

Lemma 4.2. *Suppose that $F = F_\gamma$ for some $\gamma > 1$. Then for any $\nu \in (0,1)$ with $\nu > 2 - \gamma$, a.s., for all but finitely many $n \in \mathbb{N}$, (2.8) holds.*

Proof. Let $\gamma > 1$ and $\nu > 2 - \gamma$. By Lemma 4.1, $\mathbb{P}[E(j,\nu)] = O(j^{-2})$. Hence, by the Borel–Cantelli lemma, for only finitely many $j \in \mathbb{N}$ does the vertex j connect to a vertex $i < j$ with $\rho(X_i, X_j) \geq \beta(j,\nu)$. It follows that there exists some finite random variable $D_\nu = 1 + \sum_{j=1}^\infty \mathbf{1}(E(j,\nu))$ such that, for all $n \in \mathbb{N}$ and all $i \in \{0,1,\ldots,n\}$,

$$\deg_n(i) \leq D_\nu + \sum_{j=i+1}^n \xi_{i,j},$$

where we set $\xi_{i,j} := \mathbf{1}\{\rho(X_j, X_i) \leq \beta(j,\nu)\}$ for $i \neq j$ and $\xi_{i,i} := 0$. Hence

$$\max_{0 \leq i \leq n} \deg_n(i) \leq D_\nu + \max_{0 \leq i \leq n} \sum_{j=1}^n \xi_{i,j}. \tag{4.5}$$

For fixed i, conditional on X_i, the $n-1$ terms $\xi_{i,j}$ with $j \neq i$ in the sum on the right-hand side of (4.5) are independent and $\{0,1\}$-valued, and a suitable version of Talagrand's inequality (see e.g. [14, p. 81]) will show that their sum is concentrated around its mean (in fact, we only need an upper bound here). Specifically, for $n \in \mathbb{N}$,

$$\mathbb{E} \sum_{j=1}^{n} \xi_{i,j} = \sum_{j=1}^{n} \mathbb{P}[X_i \in B(X_j; \beta(j,\nu))] = \Theta\left(\sum_{j=1}^{n} \beta(j,\nu)^d \right), \tag{4.6}$$

uniformly for $i \in \{1,\ldots,n\}$, where the implicit constants depend on S, λ_0 and λ_1 (we use Lemma 3.1 here). We claim that

$$\sum_{j=1}^{n} \beta(j,\nu)^d = \exp\{d(\log n)^\nu (1+o(1))\}. \tag{4.7}$$

To verify (4.7), we combine the upper and lower bounds

$$\begin{aligned} \sum_{j=1}^{n} \beta(j,\nu)^d &\leq \sum_{j=1}^{n} \frac{1}{j} \exp\{d(\log n)^\nu\} \leq (1+\log n)\exp\{d(\log n)^\nu\}, \text{ and} \\ \sum_{j=1}^{n} \beta(j,\nu)^d &\geq \sum_{j=\lceil n/2 \rceil}^{n} \frac{1}{n} \exp\{d(\log(n/2))^\nu\} \geq \frac{1}{2} \exp\{d(\log(n/2))^\nu\}. \end{aligned}$$

From (4.6) and (4.7), we have $\mathbb{E}\sum_{j=1}^{n} \xi_{i,j} = \exp\{d(\log n)^\nu(1+o(1))\}$. Talagrand's inequality implies that for all n,

$$\max_{0 \leq i \leq n} \mathbb{P}\left[\sum_{j=1}^{n} \xi_{i,j} > \exp\{2d(\log n)^\nu\} \right] \leq O(\exp\{-\mathrm{e}^{d(\log n)^\nu}\}),$$

which is $O(n^{-3})$, say, so that Boole's inequality yields

$$\mathbb{P}\left[\max_{0 \leq i \leq n} \sum_{j=1}^{n} \xi_{i,j} > \exp\{2d(\log n)^\nu\} \right] = O(n^{-2}).$$

Now another application of the Borel–Cantelli lemma together with (4.5) completes the proof of the lemma, noting that $\nu > 2-\gamma$ was arbitrary.

The main step remaining in the proof of Theorem 2.2 is the following.

Lemma 4.3. *Suppose that $F = F_\gamma$ for $\gamma > 3/2$. Then $\mathbb{P}[v_n \neq \eta_1(n)] \to 0$ as $n \to \infty$.*

Proof. Take a sequence $\theta_n > 0$ with $\theta_n = o(n^{-1/d})$, and, given X_n and Z_n, define the shells $A_n := B(X_n; Z_n + \theta_n) \setminus B(X_n; Z_n)$. Let $a_n := \#(A_n \cap \mathcal{X}_{n-1} \setminus \{X_{\eta_1(n)}\})$, the number of predecessors to X_n, other than its on-line nearest neighbour, inside A_n.

Conditional on X_n and Z_n, the points of $\mathcal{X}_{n-1} \setminus \{X_{\eta_1(n)}\}$ are independent and identically distributed on $S \setminus B(X_n; Z_n)$ with conditional distribution given for measurable $\Gamma \subseteq S \setminus B(X_n; Z_n)$ by $\mathbb{P}[\cdot \in \Gamma] = \int_\Gamma g_n(x) \mathrm{d}x$, where

$$g_n(x) = \frac{f(x)}{\mathbb{P}[X_0 \in S \setminus B(X_n; Z_n) \mid X_n, Z_n]}.$$

Note that, a.s.,

$$\mathbb{P}[X_0 \in S \setminus B(X_n; Z_n) \mid X_n, Z_n] = 1 - \int_{S \cap B(X_n; Z_n)} f(x) \mathrm{d}x \geq 1 - \lambda_1 \omega_d Z_n^d \geq \frac{1}{2},$$

provided $Z_n \leq (2\lambda_1 \omega_d)^{-1/d}$. Moreover, $S \cap A_n$ has volume bounded above by

$$\omega_d (Z_n + \theta_n)^d - \omega_d Z_n^d \leq C_d \theta_n (\theta_n^{d-1} + Z_n^{d-1}),$$

for some $C_d < \infty$ depending only on d. Hence, conditional on X_n and Z_n, each of the $n-1$ points $X_0, \ldots, X_{n-1}$, excluding $X_{\eta_1(n)}$, lands in A_n with probability at most

$$\frac{\int_{S \cap A_n} f(x) \mathrm{d}x}{\mathbb{P}[X_0 \in S \setminus B(X_n; Z_n) \mid X_n, Z_n]} \leq 2\lambda_1 C_d \theta_n (\theta_n^{d-1} + Z_n^{d-1}) + \mathbf{1}\{Z_n > (2\lambda_1 \omega_d)^{-1/d}\}.$$

It follows that

$$\mathbb{E}[a_n \mid Z_n] \leq 2\lambda_1 C_d n \theta_n (\theta_n^{d-1} + Z_n^{d-1}) + n \mathbf{1}\{Z_n > (2\lambda_1 \omega_d)^{-1/d}\}.$$

Taking expectations and using (3.1) we have $n\mathbb{P}[Z_n > (2\lambda_1 \omega_d)^{-1/d}] = o(1)$, while, for any $\alpha > 0$, by another application of (3.1), for some $C < \infty$,

$$\mathbb{E}[Z_n^\alpha] = \int_0^\infty \mathbb{P}[Z_n > r^{1/\alpha}] \mathrm{d}r \leq \int_0^\infty \exp\{-Cnr^{d/\alpha}\} \mathrm{d}r,$$

which gives $\mathbb{E}[Z_n^\alpha] = O(n^{-\alpha/d})$. Hence $\mathbb{E}[a_n] = O(\theta_n^d n) + O(\theta_n n^{1/d}) + o(1)$, which is $o(1)$ provided $\theta_n = o(n^{-1/d})$, so that, by Markov's inequality, $\mathbb{P}[a_n > 0] \leq \mathbb{E}[a_n] = o(1)$.

Now we condition on the whole of $\mathcal{F}_n$. Again take $\beta(n, \nu) = n^{-1/d} \exp\{(\log n)^\nu\}$. Let E_n' denote the event that X_n is joined to a point outside $B(X_n; Z_n + \theta_n)$:

$$E_n' := \{\rho(X_{v_n}, X_n) \geq Z_n + \theta_n\}.$$

Also, for a constant $b > 1$ (which later we will choose to be large), set

$$E_n'' := \{Z_n \leq b^{-1} n^{-1/d}\} \cup \{Z_n \geq b n^{-1/d}\}.$$

Finally, define the event (for another constant C to be chosen later)

$$E_n''' := \{\#\left(\mathcal{X}_{n-1} \cap B(X_n; \beta(n,\nu))\right) \geq C \exp\{d(\log n)^\nu\}\}.$$

The ball $B(X_n; \beta(n,\nu))$ has volume bounded above by $\omega_d n^{-1} \exp\{d(\log n)^\nu\}$. The events $\{X_j \in B(X_n; \beta(n,\nu))\}$, $0 \leq j \leq n-1$ are independent each with probability at most $\lambda_1 \omega_d n^{-1} \exp\{d(\log n)^\nu\}$, so $\#\left(\mathcal{X}_{n-1} \cap B(X_n; \beta(n,\nu))\right)$ is stochastically dominated by a binomial $(n, \lambda_1 \omega_d n^{-1} \exp\{d(\log n)^\nu\})$ random variable. Standard binomial tail bounds show that, for an appropriate $C < \infty$, $\mathbb{P}[E_n'''] = o(1)$.

Since $\{a_n = 0\} \cap (E_n')^{\mathrm{c}}$ implies that X_n is joined to its on-line nearest neighbour,

$$\begin{aligned} \mathbb{P}[v_n \neq \eta_1(n) \mid \mathcal{F}_n] \leq{} & \mathbb{P}[E_n' \mid \mathcal{F}_n] \mathbf{1}(\{a_n = 0\} \cap (E_n'')^{\mathrm{c}} \cap (E_n''')^{\mathrm{c}}) \\ & + \mathbf{1}\{a_n > 0\} + \mathbf{1}(E_n'') + \mathbf{1}(E_n'''). \end{aligned} \tag{4.8}$$

For any $\varepsilon > 0$, Lemma 3.2 shows that we can choose b and n_0 sufficiently large so that $\mathbb{P}[E_n''] < \varepsilon$ for all $n \geq n_0$. We have already seen that $\mathbb{P}[a_n > 0] = o(1)$ and $\mathbb{P}[E_n'''] = o(1)$. We also claim that

$$\mathbb{P}[E_n' \mid \mathcal{F}_n] \mathbf{1}(\{a_n = 0\} \cap (E_n'')^{\mathrm{c}} \cap (E_n''')^{\mathrm{c}}) = o(1), \text{ a.s.} \tag{4.9}$$

The bounded convergence theorem implies that the expectation of this last quantity is also $o(1)$, so taking expectations in (4.8) we see that for any $\varepsilon > 0$, we may choose b such that $\limsup_{n\to\infty} \mathbb{P}[v_n \neq \eta_1(n)] \leq \varepsilon$. This gives the statement in the lemma.

It remains to prove the claim (4.9). First we note that

$$D_{n-1}(X_n) \geq \deg_{n-1}(\eta_1(n)) F_\gamma(\rho(X_{\eta_1(n)}, X_n)) \geq F_\gamma(Z_n).$$

On the other hand, on $\{a_n = 0\}$, any alternative X_j to $X_{\eta_1(n)}$ among $\mathcal{X}_{n-1}$ is at distance at least $Z_n + \theta_n$ from X_n, so that for $j \neq \eta_1(n)$,

$$\deg_{n-1}(j) F_\gamma(\rho(X_j, X_n)) \leq \exp\{(\log n)^\nu\} F_\gamma(Z_n + \theta_n), \text{ a.s.},$$

for all n large enough, by Lemma 4.2, provided $\nu \in (0,1)$ with $\nu > 2 - \gamma$.

On $(E_n''')^{\mathrm{c}} \cap \{a_n = 0\}$, the contribution of points inside $B(X_n; \beta(n,\nu))$, other than $X_{\eta_1(n)}$, to $D_{n-1}(X_n)$ is bounded above by $C \exp\{2d(\log n)^\nu\} F_\gamma(Z_n + \theta_n)$, since there are at most $O(\exp\{d(\log n)^\nu\})$ of these points, their degrees are at most $O(\exp\{(\log n)^\nu\})$, a.s., by Lemma 4.2, and they are all at distance at least $Z_n + \theta_n$ from X_n. Moreover, similarly to as in the proof of Lemma 4.1, the contribution to $D_{n-1}(X_n)$ from any points outside $B(X_n;\beta(n,\nu))$ is at most $n^2 F_\gamma(\beta(n,\nu))$.

So from (3.6) we have, on $\{a_n = 0\} \cap (E_n''')^{\mathrm{c}}$, for all n large enough,

$$\mathbb{P}[E_n' \mid \mathcal{F}_n] \leq \frac{C\exp\{2d(\log n)^\nu\} F_\gamma(Z_n + \theta_n) + n^2 F_\gamma(\beta(n,\nu))}{F_\gamma(Z_n)}.$$

Here, similarly to (4.4),

$$\frac{n^2 F_\gamma(\beta(n,\nu))}{F_\gamma(Z_n)} = O(\exp\{-c(\log n)^{\gamma+\nu-1}\}),$$

for some $c > 0$, as long as $\nu > 2 - \gamma$. Also we have that, on $(E_n'')^{\mathrm{c}}$,

$$\begin{aligned}\frac{F_\gamma(Z_n + \theta_n)}{F_\gamma(Z_n)} &= \exp\left\{(\log(1/Z_n))^\gamma \left(\left(1 + \frac{\log(1+(\theta_n/Z_n))}{\log Z_n}\right)^\gamma - 1\right)\right\} \\ &= \exp\left\{-c'(\log n)^{\gamma-1} n^{1/d} \theta_n (1+o(1))\right\},\end{aligned}$$

for some $c' > 0$, if $\theta_n = o(n^{-1/d})$. In particular, for $\gamma - 1 > \nu$, we can choose $\theta_n = n^{1/d}(\log n)^{1-\gamma+\nu+\varepsilon}$ for some $\varepsilon > 0$ and $1 - \gamma + \nu + \varepsilon < 0$. The constraints $\gamma - 1 > \nu$ and $\nu > 2 - \gamma$ entail $\gamma > 3/2$. With this choice of θ_n, we thus verify (4.9).

Now we can complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Part (i) is Lemma 4.2. It remains to prove part (ii). Let $R_n = \sum_{i=1}^n \mathbf{1}\{v_i \neq \eta_1(i)\}$. Then, by Lemma 4.3, $\mathbb{E}R_n = o(n)$, which gives (2.9). Moreover, the obvious coupling is such that, given $\mathcal{X}_n$ and G_n, one can transform the GPA graph G_n into the ONG on the same vertex sequence by the reassignment of the endpoint with smaller index of R_n edges, so affecting the degrees of at most $2R_n$ vertices. Hence, with this coupling, for any $k \in \mathbb{N}$, $n^{-1}\left|N_n^{\mathrm{GPA}}(k) - N_n^{\mathrm{ONG}}(k)\right| \leq 2n^{-1}R_n$, which tends to 0 in L^1. Now the L^1 limit statement in (2.3) yields (2.10).

5. Proofs for power-law attractiveness

5.1. Rejecting on-line nearest-neighbours

We introduce some notation on Voronoi cells that will also be used in Section 6. Let $\mathcal{V}_n(i)$ denote the (bounded) Voronoi cell of X_i with respect to $\mathcal{X}_n$ in S, i.e.,

$$\mathcal{V}_n(i) := \{x \in S : \rho(x, X_i) < \min\{\rho(x, X_j) : 0 \leq j \leq n,\, j \neq i\}\}. \tag{5.1}$$

We need an elementary result showing that Voronoi cells are unlikely to be very small.

Lemma 5.1. *For any $z > 0$, with $\delta_S > 0$ the constant in Lemma 3.1,*

$$\mathbb{P}[|\mathcal{V}_n(i)| < z] \leq 2^d \lambda_1 \delta_S^{-1} nz. \tag{5.2}$$

Proof. We follow the idea from [1, p. 311]. If no X_j with $0 \leq j \leq n$, $j \neq i$ lies in $B(X_i; r)$, then $S \cap B(X_i; r/2) \subseteq \mathcal{V}_n(i)$ and hence $|\mathcal{V}_n(i)| \geq \delta_S \omega_d (r/2)^d$ by Lemma 3.1. That is, $\mathbb{P}[|\mathcal{V}_n(i)| \geq \delta_S \omega_d (r/2)^d] \geq \mathbb{P}[\mathcal{X}_n \cap B(X_i; r) = \{X_i\}]$. Complementation then shows that $|\mathcal{V}_n(i)| < z$ ($z > 0$) implies that at least one of n points X_j falls in $B(X_i; 2z^{1/d}/(\omega_d \delta_S)^{1/d})$. Now (5.2) follows from an application of Boole's inequality:

$$\mathbb{P}[|\mathcal{V}_n(i)| < z] \leq n\mathbb{P}[X_j \in B(X_i; 2z^{1/d}/(\omega_d \delta_S)^{1/d})] \leq 2^d \delta_S^{-1} \lambda_1 nz.$$

Take $F(r) = r^{-s}$ for $s \in (0, \infty)$. To prove Theorem 2.3, we consider the event $\{v_n \neq \eta_1(n)\}$ that X_n is joined to a point *other than* its nearest neighbour.

Proof of Theorem 2.3. Extending the notation of (2.2), for $\ell \in \mathbb{N}$ we let $\eta_\ell(n)$ be the index of the ℓth nearest neighbour of X_n among $\mathcal{X}_{n-1}$. Again set $Z_n = \rho(X_n, X_{\eta_1(n)})$ and $W_n = \rho(X_n, X_{\eta_2(n)})$. Then by (3.6),

$$\frac{\mathbb{P}[v_n \neq \eta_1(n) \mid \mathcal{F}_n]}{\mathbb{P}[v_n = \eta_1(n) \mid \mathcal{F}_n]} \geq \frac{\mathbb{P}[v_n = \eta_2(n) \mid \mathcal{F}_n]}{\mathbb{P}[v_n = \eta_1(n) \mid \mathcal{F}_n]} \geq \frac{F(W_n)}{\deg_{n-1}(\eta_1(n)) F(Z_n)}.$$

Re-arranging and using the fact that $F(r) = r^{-s}$, we obtain

$$\mathbb{P}[v_n \neq \eta_1(n) \mid \mathcal{F}_n] \geq \frac{1}{1 + \deg_{n-1}(\eta_1(n))(W_n/Z_n)^s} \geq \frac{(Z_n/W_n)^s}{2\deg_{n-1}(\eta_1(n))}. \tag{5.3}$$

Then (2.11) will follow from (5.3) together with the following two claims: first, there exist constants $k_0 \in \mathbb{N}$ and $\theta_0 \in (0, 1)$ such that

$$\liminf_{n \to \infty} \mathbb{P}[\deg_n(\eta_1(n+1)) \leq k_0] \geq 2\theta_0, \tag{5.4}$$

and second, that for any $\theta > 0$ there exist constants $c, C \in (0, \infty)$ such that,

$$\mathbb{P}[Z_n \geq cn^{-1/d}] \geq 1 - (\theta/3), \quad \text{and} \quad \mathbb{P}[W_n \leq Cn^{-1/d}] \geq 1 - (\theta/3), \tag{5.5}$$

for all n sufficiently large. Indeed, it follows from (5.5) that $\mathbb{P}[Z_n/W_n \geq c/C] \geq 1 - (2\theta_0/3)$ for suitable choice of c and C, so that, by (5.3) and (5.4), $\mathbb{P}[v_n \neq \eta_1(n) \mid \mathcal{F}_n] \geq \frac{1}{2k_0}(c/C)^s$ with probability at least $\theta_0/3$ for all n sufficiently large. Then, taking expectations, we obtain (2.11). Thus it remains to prove the claims (5.4) and (5.5).

The idea behind (5.4) is that a large proportion of vertices have degrees bounded above by some k_0, and the union of the corresponding Voronoi cells will have volume bounded uniformly below in expectation, so that X_{n+1} will have such a vertex as its nearest neighbour with strictly positive probability. We formalize this idea.

With $I_n(k) := \{i \in \{0, \dots, n\} : \deg_n(i) \leq k\}$, we have $\# I_n(k) = n+1-N_n^{\mathrm{GPA}}(k+1)$. Taking $k_0 = 9$, we obtain from (3.5) that $\# I_n(k_0) \geq 4n/5$ for all n. Each vertex $i \in I_n(k_0)$ is associated with a Voronoi cell $\mathcal{V}_n(i)$.

Let $\Lambda_n(r) = \#\{i \in \{0, \dots, n\} : |\mathcal{V}_n(i)| \geq r/n\}$. Then

$$\mathbb{E}[\Lambda_n(r)] = \sum_{i=0}^{n} \mathbb{P}[|\mathcal{V}_n(i)| \geq r/n] = (n+1)\mathbb{P}[|\mathcal{V}_n(i)| \geq r/n],$$

by exchangeability. Here, by (5.2), $\mathbb{P}[|\mathcal{V}_n(i)| \geq r/n] \geq 1 - 2^d \lambda_1 \delta_S^{-1} r$. Hence we can (and do) choose $r = r_0$ sufficiently small so that $\mathbb{E}[\Lambda_n(r_0)] \geq 9n/10$, say. Then, by Markov's inequality and the fact that $\Lambda_n(r_0) \leq 1 + n$,

$$\mathbb{P}[\Lambda_n(r_0) \leq n/2] \leq \mathbb{P}[n + 1 - \Lambda_n(r_0) \geq n/2] \leq \frac{1 + (n/10)}{n/2} \leq 1/4, \text{ for all } n \geq 40.$$

So $\mathbb{P}[\Lambda_n(r_0) \geq n/2] \geq 3/4$ for all $n \geq 40$. On $\{\Lambda_n(r_0) \geq n/2\}$, since $\# I_n(k_0) \geq 4n/5$, at least $3n/10$ vertices in $I_n(k_0)$ have Voronoi cells of volume at least r_0/n, so

$$\mathbb{P}\left[\left| \bigcup_{i \in I_n(k_0)} \mathcal{V}_n(i) \right| \geq 3r_0/10 \right] \geq 3/4, \tag{5.6}$$

for all n sufficiently large. Hence

$$\mathbb{P}\left[\deg_n(\eta_1(n+1)) \leq k_0\right] \geq \mathbb{P}\left[X_{n+1} \in \bigcup_{i \in I_n(k_0)} \mathcal{V}_n(i) \right] \geq \lambda_0 \mathbb{E}\left[\left| \bigcup_{i \in I_n(k_0)} \mathcal{V}_n(i) \right| \right],$$

which with (5.6) gives (5.4), for $2\theta_0 = 9r_0\lambda_0/40 > 0$.

Finally, (5.5) can be verified by a similar argument to Lemma 3.2.

5.2. Stretched exponential degree estimates

Throughout this section we take $F(r) = r^{-s}$ for $s > d$. By (3.6), for $0 \leq i \leq n-1$,

$$\mathbb{P}[v_n = i \mid \mathcal{F}_n] = \frac{\deg_{n-1}(i)\rho(X_i, X_n)^{-s}}{\sum_{j=0}^{n-1} \deg_{n-1}(j)\rho(X_j, X_n)^{-s}}.$$

Define, for any $x \in S$,

$$\zeta_{n-1}(x) := n^{-s/d} \sum_{j=0}^{n-1} \rho(X_j, x)^{-s}. \tag{5.7}$$

Then we can write

$$\mathbb{P}[v_n = i \mid \mathcal{F}_n] \leq \frac{\deg_{n-1}(i)\rho(X_i, X_n)^{-s}}{n^{s/d}\zeta_{n-1}(X_n)}. \tag{5.8}$$

The next result gives an estimate for the probability that $\zeta_{n-1}(X_n)$ is small.

Lemma 5.2. *There exist constants $C_0 < \infty$ and $u_0 > 0$ such that, for all $t > 0$,*

$$\limsup_{n\to\infty} \mathbb{P}[\zeta_{n-1}(X_n) \leq t] \leq C_0 \exp\{-u_0 t^{-d/(s-d)}\}. \tag{5.9}$$

Proof. First, for fixed $x \in S$, we have the tail estimate, for $r > 0$,

$$\mathbb{P}[\rho(X_j, x)^{-s} > r] = \mathbb{P}[X_j \in B(x; r^{-1/s})] \geq \lambda_0 \delta_S \omega_d r^{-d/s},$$

using (2.1) and the lower bound in (3.4). Hence $\zeta_{n-1}(x)$ stochastically dominates $\zeta_{n-1} := n^{-s/d} \sum_{j=0}^{n-1} \xi_j$, where $\xi_j \in \mathbb{R}_+$ are i.i.d. with $\mathbb{P}[\xi_j > r] = \lambda_0 \delta_S \omega_d r^{-d/s}$. Here ζ_{n-1} converges in distribution as $n \to \infty$ to a random variable ζ with a positive stable law of index $d/s \in (0, 1)$. Hence, for all $x \in S$ and any $t > 0$,

$$\limsup_{n\to\infty} \mathbb{P}[\zeta_{n-1}(x) \leq t] \leq \lim_{n\to\infty} \mathbb{P}[\zeta_{n-1} \leq t] = \mathbb{P}[\zeta \leq t].$$

Given that ζ is a random variable with a positive stable law with index $\alpha \in (0, 1)$, for $p > 0$ the random variable ζ^{-p} satisfies $\mathbb{E}[\exp(u\zeta^{-p})] < \infty$ for $u \geq 0$ in a neighbourhood of zero, provided $p \leq \frac{\alpha}{1-\alpha}$: see e.g. the proof of Lemma 1 in [3]. Hence there exist $u_0 > 0$ and $C_0 < \infty$ such that, for $p = \frac{d}{s-d} > 0$, $\mathbb{E}[\exp(u_0\zeta^{-p})] \leq C_0$. Thus $\mathbb{P}[\zeta \leq t] = \mathbb{P}[\exp(u_0\zeta^{-p}) \geq \exp(u_0 t^{-p})]$, and the result now follows from Markov's inequality.

The next result is a conditional version of (5.9), given $\mathcal{X}_{n-1}$. The proof uses a concentration argument based on independently 'resampling' sites. Let $X_0', X_1', \ldots$ be an independent copy of the sequence $X_0, X_1, \ldots$. For $0 \leq i \leq n$, let $\mathcal{X}_n^i = (X_0, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$, the sites $\mathcal{X}_n$ but with X_i replaced by X_i'.

Lemma 5.3. *There exist constants $C_1 < \infty$ and $u_1 > 0$ such that, for any $t > 0$, a.s.,*

$$\limsup_{n\to\infty} \mathbb{P}[\zeta_{n-1}(X_n) \leq t \mid \mathcal{X}_{n-1}] \leq C_1 \exp\{-u_1 t^{-d/(s-d)}\}.$$

Proof. We approximate the indicator function $\mathbf{1}_{[0,t]}$ by $\chi_t^n : \mathbb{R}_+ \to [0, 1]$ defined by

$$\chi_t^n(x) := \begin{cases} 1 & \text{if } x \leq t \\ 1 - (x-t)n^{\delta} & \text{if } t \leq x \leq t + n^{-\delta} \\ 0 & \text{if } x \geq t + n^{-\delta}, \end{cases}$$

where $\delta > 0$ is a constant to be specified later. Then

$$\begin{aligned}\mathbb{P}[\zeta_{n-1}(X_n) \le t \mid \mathcal{X}_{n-1}] &= \mathbb{E}[\mathbf{1}_{[0,t]}(\zeta_{n-1}(X_n)) \mid \mathcal{X}_{n-1}] \\ &\le \mathbb{E}[\chi_t^n(\zeta_{n-1}(X_n)) \mid \mathcal{X}_{n-1}],\end{aligned}$$

and χ_t^n has the one-sided Lipschitz property $\chi_t^n(r) - \chi_t^n(s) \le n^\delta (s-r)^+$. Now

$$\mathbb{E}[\chi_t^n(\zeta_{n-1}(X_n)) \mid \mathcal{X}_{n-1}] = \int_S f(x)\chi_t^n(\zeta_{n-1}(x))\mathrm{d}x = \phi(\mathcal{X}_{n-1})$$

for some measurable $\phi : S^n \to [0,1]$. To obtain a concentration result for $\phi(\mathcal{X}_{n-1})$, we estimate $\phi(\mathcal{X}_{n-1}^i) - \phi(\mathcal{X}_{n-1})$. We introduce the notation

$$\zeta_{n-1}^i(x) = \zeta_{n-1}(x) + n^{-s/d}\left(\rho(X_i', x)^{-s} - \rho(X_i, x)^{-s}\right), \tag{5.10}$$

the change in the quantity given by (5.7) on resampling X_i. Then, for $r_n > 0$,

$$\begin{aligned}\phi(\mathcal{X}_{n-1}^i) - \phi(\mathcal{X}_{n-1}) &\le \int_{B(X_i;r_n)} f(x)\mathrm{d}x + \int_{S\setminus B(X_i;r_n)} f(x)\left(\chi_t^n(\zeta_{n-1}^i(x)) - \chi_t^n(\zeta_{n-1}(x))\right)\mathrm{d}x \\ &\le \lambda_1\omega_d r_n^d + \int_{S\setminus B(X_i;r_n)} n^\delta f(x)\left(\zeta_{n-1}(x) - \zeta_{n-1}^i(x)\right)^+ \mathrm{d}x,\end{aligned}$$

using the one-sided Lipschitz property of χ_t^n. Now, by (5.10),

$$\left(\zeta_{n-1}(x) - \zeta_{n-1}^i(x)\right)^+ \le n^{-s/d}\rho(x, X_i)^{-s} \le n^{-s/d} r_n^{-s},$$

provided $x \notin B(X_i; r_n)$. So we obtain $\phi(\mathcal{X}_{n-1}^i) - \phi(\mathcal{X}_{n-1}) \le \lambda_1\omega_d r_n^d + n^\delta n^{-s/d} r_n^{-s}$. Since $s > d$, we may choose $\delta > 0$ such that $(s/d) - \delta > 1$. Take $r_n = n^{-\nu}$ where $\nu = \frac{(s/d)-\delta}{s+d} > 0$. Then we have that, for some constant $C < \infty$,

$$\phi(\mathcal{X}_{n-1}^i) - \phi(\mathcal{X}_{n-1}) \le Cn^{-\frac{d((s/d)-\delta)}{s+d}} \le Cn^{-\frac{d}{s+d}}.$$

A version of Talagrand's inequality [13, Theorem 4.5] yields, for some $c_1 > 0$,

$$\mathbb{P}[|\phi(\mathcal{X}_{n-1}) - m_{n-1}| \ge r] \le 4\exp\left\{-c_1 n^{\frac{2d}{s+d}} r^2\right\}, \text{ for all } r > 0, \tag{5.11}$$

where m_{n-1} is a median of $\phi(\mathcal{X}_{n-1})$. In turn, (5.11) implies, by Lemma 4.6 of [13], that $|m_{n-1} - \mathbb{E}\phi(\mathcal{X}_{n-1})| \le c_2 n^{-\frac{d}{s+d}}$ for some $c_2 < \infty$. Here $\mathbb{E}\phi(\mathcal{X}_{n-1}) = \mathbb{E}[\chi_t^n(\zeta_{n-1}(X_n))] \ge \mathbb{P}[\zeta_{n-1}(X_n) \le t]$, which for a fixed $t > 0$ is bounded below uniformly in n, as can be proved using an analogous argument to the proof of Lemma 5.2, this time using the upper bound in (3.4). It follows that, for some $c_3 > 0$,

$$\mathbb{P}[\phi(\mathcal{X}_{n-1}) \ge 2\mathbb{E}\phi(\mathcal{X}_{n-1})] \le 4\exp\left\{-c_3 n^{\frac{2d}{s+d}}\right\}. \tag{5.12}$$

The right-hand side of (5.12) is summable in n, so the Borel–Cantelli lemma shows

$$\mathbb{P}[\zeta_{n-1}(X_n) \leq t \mid \mathcal{X}_{n-1}] \leq \phi(\mathcal{X}_{n-1}) \leq 2\mathbb{E}\phi(\mathcal{X}_{n-1}), \text{ a.s.,}$$

for all but finitely many n. Here, for $t > 0$,

$$\mathbb{E}\phi(\mathcal{X}_{n-1}) \leq \mathbb{P}[\zeta_{n-1}(X_n) \leq t + n^{-\delta}] \leq \mathbb{P}[\zeta_{n-1}(X_n) \leq 2t]$$

for all n large enough. Now the statement follows from (5.9).

Choosing $t = k^{-\gamma(s-d)/d}$ with $\gamma \in (0,1)$ in Lemma 5.3, we obtain the key estimate

$$\limsup_{n\to\infty} \mathbb{P}[\zeta_{n-1}(X_n) \leq k^{-\gamma(s-d)/d} \mid \mathcal{X}_{n-1}] \leq C_1 \exp\{-u_1 k^\gamma\}, \text{ a.s.} \tag{5.13}$$

In what follows, $C_2, C_3, \ldots$ represent constants not depending on n or k. We have,

$$\begin{aligned}\mathbb{P}[v_n = i,\, \zeta_{n-1}(X_n) > t \mid \tilde{\mathcal{F}}_{n-1}] &\leq \mathbb{P}[\rho(X_i, X_n) \leq Bn^{-1/d} \mid \tilde{\mathcal{F}}_{n-1}] \\ &\quad + \mathbb{P}[v_n = i,\, \rho(X_i, X_n) > Bn^{-1/d},\, \zeta_{n-1}(X_n) > t \mid \tilde{\mathcal{F}}_{n-1}],\end{aligned} \tag{5.14}$$

for any $B > 0$ and any $t > 0$. The first term on the right-hand side of (5.14) is at most $C_2 B^d n^{-1}$, and the second term, by (5.8), is bounded above by

$$\frac{\deg_{n-1}(i)}{tn^{s/d}} \int_S f(x) \rho(X_i, x)^{-s} \mathbf{1}\{\rho(X_i, x) > Bn^{-1/d}\} \mathrm{d}x.$$

For $s > d$, the latter integral is bounded above by

$$C_3 \int_{Bn^{-1/d}}^{\infty} \rho^{-s} \rho^{d-1} \mathrm{d}\rho = C_4 B^{d-s} n^{(s/d)-1}.$$

Hence we obtain from (5.14) that

$$\mathbb{P}[v_n = i,\, \zeta_{n-1}(X_n) > t \mid \tilde{\mathcal{F}}_{n-1}] \leq n^{-1} \left(C_2 B^d + \frac{C_4}{t} B^{d-s} \deg_{n-1}(i) \right). \tag{5.15}$$

For ease of notation, let $q_k^{(n)}$ be the proportion of vertices of G_n with degree at least k, so that $q_k^{(n)} := (n+1)^{-1} N_n^{\mathrm{GPA}}(k)$. Then the proportion of vertices of G_n with degree k is equal to $q_k^{(n)} - q_{k+1}^{(n)}$, so that (5.15) yields

$$\begin{aligned}\mathbb{P}[\deg_{n-1}(v_n) = k,\, \zeta_{n-1}(X_n) > t \mid \tilde{\mathcal{F}}_{n-1}] &= \sum_{i:\deg_{n-1}(i)=k} \mathbb{P}[v_n = i,\, \zeta_{n-1}(X_n) > t \mid \tilde{\mathcal{F}}_{n-1}] \\ &\leq \left(q_k^{(n-1)} - q_{k+1}^{(n-1)} \right) \left(C_2 B^d + \frac{C_4}{t} B^{d-s} k \right).\end{aligned}$$

We take $t = k^{-\gamma(s-d)/d}$ for $\gamma \in (0,1)$, and choose $B = k^{(\gamma/d)+(1/s)(1-\gamma)}$ to get

$$\mathbb{P}[\deg_{n-1}(v_n) = k,\, \zeta_{n-1}(X_n) > k^{-\gamma(s-d)/d} \mid \tilde{\mathcal{F}}_{n-1}] \leq C_5 \Big(q_k^{(n-1)} - q_{k+1}^{(n-1)} \Big) k^{\gamma+(d/s)(1-\gamma)}.$$

Now incorporating the case where $\zeta_{n-1}(X_n)$ is small, using (5.13), gives, a.s.,

$$\mathbb{P}[\deg_{n-1}(v_n) = k \mid \tilde{\mathcal{F}}_{n-1}] \leq C_6 \mathrm{e}^{-u_1 k^\gamma} + C_5 \left(q_k^{(n-1)} - q_{k+1}^{(n-1)} \right) k^\beta, \tag{5.16}$$

for all n sufficiently large, where we have set $\beta = \gamma + (d/s)(1-\gamma)$. For any k, between times $n-1$ and n, the number of vertices of degree at least k either stays the same, or increases by exactly one if and only if $\deg_{n-1}(v_n) = k-1$, so that $\deg_n(v_n) = k$. Thus

$$\mathbb{E}[q_{k+1}^{(n)} \mid \tilde{\mathcal{F}}_{n-1}] - q_{k+1}^{(n-1)} = \frac{1}{n+1} \left(n q_{k+1}^{(n-1)} + \mathbb{P}[\deg_{n-1}(v_n) = k \mid \tilde{\mathcal{F}}_{n-1}] \right) - q_{k+1}^{(n-1)},$$

and we may express (5.16) as

$$\begin{aligned} \mathbb{E}[q_{k+1}^{(n)} \mid \tilde{\mathcal{F}}_{n-1}] - q_{k+1}^{(n-1)} &\leq \frac{1}{n+1} \left(C_6 \mathrm{e}^{-u_1 k^\gamma} + C_5 \left(q_k^{(n-1)} - q_{k+1}^{(n-1)} \right) k^\beta - q_{k+1}^{(n-1)} \right) \\ &= \frac{1}{n+1} \left(C_6 \mathrm{e}^{-u_1 k^\gamma} + q_k^{(n-1)} C_5 k^\beta - q_{k+1}^{(n-1)} (1 + C_5 k^\beta) \right). \end{aligned} \tag{5.17}$$

If we suppose that $q_k^{(n)} \leq \tau_k$ for some τ_k and all n sufficiently large (which we can, of course, always do for $\tau_k = 1$) then (5.17) gives, for n large enough,

$$\mathbb{E}[q_{k+1}^{(n)} \mid \tilde{\mathcal{F}}_{n-1}] - q_{k+1}^{(n-1)} \leq \frac{1}{n+1} \left(C_6 \mathrm{e}^{-u_1 k^\gamma} + \tau_k C_5 k^\beta - q_{k+1}^{(n-1)} (1 + C_5 k^\beta) \right). \tag{5.18}$$

The final step in the proof of Theorem 2.4 is an analysis of (5.18) that will enable us iteratively to improve the bound τ_k; this uses the following stochastic approximation result, which is related to Lemma 2.6 of [15] and of some independent interest.

Lemma 5.4. *Let $(\mathcal{G}_n; n \in \mathbb{Z}_+)$ be a filtration. Let g be a bounded function on $\mathbb{R}_+$. For $n \in \mathbb{Z}_+$, let Y_n, r_n, ξ_n be $\mathcal{G}_n$-measurable random variables, with $Y_n \in \mathbb{R}_+$, and*

$$Y_{n+1} - Y_n \leq \gamma_n \left(g(Y_n) + \xi_{n+1} + r_n \right), \tag{5.19}$$

for constants $\gamma_n > 0$. Suppose also that

(i) $\mathbb{E}[\xi_{n+1} \mid \mathcal{G}_n] = 0$ *and* $\mathbb{E}[\xi_{n+1}^2 \mid \mathcal{G}_n] \leq C$ *for some constant* $C < \infty$*;*

(ii) $\sum_n \gamma_n = \infty$, $\sum_n \gamma_n^2 < \infty$, *and* $\sum_n \gamma_n |r_n| < \infty$ *a.s.;*

(iii) $g(y) < -\delta$ *for* $y > y_0$ *for constants* $\delta > 0$ *and* $y_0 \in \mathbb{R}_+$.

Then $\limsup_{n\to\infty} Y_n \leq y_0$, *a.s.*

Proof. Summing (5.19) we obtain $Y_n - Y_0 \leq M_n + A_n$ for any $n \in \mathbb{Z}_+$, where

$$M_n = \sum_{k=0}^{n-1} \gamma_k \xi_{k+1}, \quad \text{and} \quad A_n = \sum_{k=0}^{n-1} \gamma_k \left(g(Y_k) + r_k \right);$$

the $\mathcal{G}_n$-martingale M_n and $\mathcal{G}_{n-1}$-measurable A_n constitute the Doob decomposition of the process whose increments are the right-hand side of (5.19). By (i),

$$\mathbb{E}[M_{n+1}^2 - M_n^2 \mid \mathcal{G}_n] = \mathbb{E}[(M_{n+1} - M_n)^2 \mid \mathcal{G}_n] \leq C\gamma_n^2, \text{ a.s.},$$

which is summable, by (ii), so the increasing process associated with M_n is a.s. bounded. Hence $M_n \to M_\infty$ a.s., for some finite limit M_∞. Also, writing $R_n = \sum_{k=0}^{n-1} \gamma_k r_k$, we have $R_n \to R_\infty$ a.s. for some finite limit R_∞, by (ii). In particular, for any $\varepsilon > 0$, there exists an a.s. finite N such that,

$$\sup_{n\geq N} \sup_{m\geq 0} |M_{n+m} - M_n| \leq \varepsilon/4, \quad \text{and} \quad \sup_{n\geq N} \sup_{m\geq 0} |R_{n+m} - R_n| \leq \varepsilon/4.$$

Consider some $n \geq N$ for which $Y_n > y_0$. Let κ_n be the first time after n for which $Y_\cdot \leq y_0$. Then, by summing (5.19) again, for $m \geq 0$,

$$\begin{aligned} Y_{(n+m)\wedge\kappa_n} - Y_n &\leq M_{(n+m)\wedge\kappa_n} - M_n + R_{(n+m)\wedge\kappa_n} - R_n + \sum_{k=n}^{(n+m)\wedge\kappa_n - 1} \gamma_k g(Y_k) \\ &\leq \frac{\varepsilon}{2} - \delta \sum_{k=n}^{(n+m)\wedge\kappa_n - 1} \gamma_k. \end{aligned}$$

In particular, on $\{\kappa_n = \infty\}$, letting $m \to \infty$ the left-hand side of the last display remains bounded below by $-Y_n$ while the right-hand side tends to $-\infty$, by (ii); hence $\kappa_n < \infty$ a.s., and the process returns to the interval $[0, y_0]$ without exceeding $Y_n + \varepsilon$. Moreover, $Y_{n+1} - Y_n \leq \frac{\varepsilon}{2} + \gamma_n g(Y_n) < \varepsilon$, for all $n \geq N$ large enough, since g is bounded and $\gamma_n \to 0$. Hence $Y_n \leq y_0$ infinitely often, and, for all but finitely many such n, any exit from $[0, y_0]$ cannot exceed $y_0 + \varepsilon$; but starting from $[y_0, y_0 + \varepsilon]$ the process returns to $[0, y_0]$ before reaching $y_0 + 2\varepsilon$. Hence $\limsup_{n\to\infty} Y_n \leq y_0 + 2\varepsilon$, a.s. Since $\varepsilon > 0$ was arbitrary, the result follows.

Proof of Theorem 2.4. Setting $\mathcal{G}_n = \tilde{\mathcal{F}}_n$, $Y_n = q_{k+1}^{(n)}$, $\gamma_n = \frac{1}{n+2}$, $r_n = 0$,

$$g(y) = C_6 \mathrm{e}^{-u_1 k^\gamma} + \tau_k C_5 k^\beta - y(1 + C_5 k^\beta), \text{ and } \xi_{n+1} = (n+2)\left(q_{k+1}^{(n+1)} - \mathbb{E}[q_{k+1}^{(n+1)} \mid \tilde{\mathcal{F}}_n] \right),$$

we apply Lemma 5.4 to (5.18). Note that, since $N_n^{\mathrm{GPA}}(k)$ is $\tilde{\mathcal{F}}_n$-measurable,

$$\begin{aligned}\xi_{n+1} &= N_{n+1}^{\mathrm{GPA}}(k+1) - \mathbb{E}[N_{n+1}^{\mathrm{GPA}}(k+1) \mid \tilde{\mathcal{F}}_n]\\ &= N_{n+1}^{\mathrm{GPA}}(k+1) - N_n^{\mathrm{GPA}}(k+1) - \mathbb{E}[N_{n+1}^{\mathrm{GPA}}(k+1) - N_n^{\mathrm{GPA}}(k+1) \mid \tilde{\mathcal{F}}_n],\end{aligned}$$

which is uniformly bounded, since $0 \leq N_{n+1}^{\mathrm{GPA}}(k) - N_n^{\mathrm{GPA}}(k) \leq 1$, a.s. Hence the conditions of Lemma 5.4 are satisfied for any

$$y_0 > \frac{C_6 \mathrm{e}^{-u_1 k^\gamma} + \tau_k C_5 k^\beta}{1 + C_5 k^\beta},$$

and we deduce that

$$\limsup_{n\to\infty} q_{k+1}^{(n)} \leq \frac{C_6 \mathrm{e}^{-u_1 k^\gamma} + \tau_k C_5 k^\beta}{1 + C_5 k^\beta}. \tag{5.20}$$

In particular, if $q_k^{(n)} \leq \tau_k$ for all but finitely many n, a.s., then (5.20) implies that $q_{k+1}^{(n)} \leq \tau_{k+1}$ for all but finitely many n, a.s., where

$$\tau_{k+1} = \frac{2C_6 \mathrm{e}^{-u_1 k^\gamma} + \tau_k C_5 k^\beta}{1 + C_5 k^\beta}. \tag{5.21}$$

Defining $\sigma_k > 0$ via $\tau_k = 2C_6 \sigma_k \mathrm{e}^{-u_1 k^\gamma}$, we obtain from (5.21) , after some algebra,

$$\sigma_{k+1} - \sigma_k = \left(1 - a_{k+1} + \frac{a_{k+1}}{1 + C_5 k^\beta}\right)(1 - \sigma_k) - (1 - a_{k+1}),$$

where $a_{k+1} := \exp\{u_1((k+1)^\gamma - k^\gamma)\} = 1 + \gamma u_1 k^{\gamma-1} + O(k^{\gamma-2})$, as $k \to \infty$.

Then, assuming that $\beta < 1 - \gamma$, it is straightforward to check that,

$$1 - a_{k+1} + \frac{a_{k+1}}{1 + C_5 k^\beta} \sim \frac{1}{C_5 k^\beta}, \text{ as } k \to \infty.$$

Hence we may apply Lemma 1 of [10] to see that $\lim_{k\to\infty} \sigma_k = 1$, provided $\beta < 1 - \gamma$, i.e., $\gamma < \frac{s-d}{2s-d}$. For any such γ, we thus obtain $\limsup_{n\to\infty} q_k^{(n)} \leq \tau_k \leq 3C_6 \mathrm{e}^{-u_1 k^\gamma}$, a.s., for all k sufficiently large, giving the almost sure statement in the theorem. Then the reverse Fatou lemma yields the statement on expectations.

6. Proofs for the on-line nearest-neighbour graph

In this section we prove Theorem 2.1. Our argument extends the 2-dimensional argument of [1, §3.1], who considered the uniform distribution on the square.

Recall the definition of the Voronoi cell $\mathcal{V}_n(i)$ from (5.1). Then

$$\mathcal{V}_{n+1}(i) = \mathcal{V}_n(i) \cap \{x \in S : \rho(x, X_i) < \rho(x, X_{n+1})\} \subseteq \mathcal{V}_n(i). \tag{6.1}$$

A key fact is provided by the following lemma, which will be used to show that the volume of a Voronoi cell associated with a vertex in the ONG shrinks, on average, by a positive fraction whenever a new vertex lands in the cell.

Lemma 6.1. *Let $R \subseteq S$ be convex, and let X be a random point in S distributed according to the probability density f satisfying* (2.1)*. For $x_0 \in R$, let $R' = \{x \in R : \rho(x, x_0) < \rho(x, X)\}$. Then there exists $\delta > 0$ not depending on R or x_0 such that*

$$\mathbb{E}[|R'| \mid X \in R] \leq (1 - \delta)\mathbb{E}[|R|].$$

Proof. Without loss of generality, take $x_0 = 0 \in R$. Partition R according to the Cartesian orthants as $R_1, \ldots, R_{2^d}$. For each j, any two points $x, y \in R_j$ have the same signs in corresponding coordinates, so $\|x-y\| \leq \|x+y\|$, and hence $(x+y)/2$ is closer to x (and to y) than to 0. Thus, given $X \in R_j$, any point x of $R_j'' := \{(X+y)/2 : y \in R_j\}$ has $\|x-X\| \leq \|x-0\|$, and, by convexity, $R_j'' \subseteq R_j$. Hence, given $X \in R_j$, $R' \subseteq R \setminus R_j''$. By construction, R_j'' is a translate of R_j scaled by a factor of $1/2$, so

$$\begin{aligned}\mathbb{E}[|R'| \mid X \in R] &\leq |R| - \sum_{j=1}^{2^d} 2^{-d}|R_j|\mathbb{P}[X \in R_j \mid X \in R] \\ &\leq |R| - 2^{-d}(\lambda_0/\lambda_1)|R|^{-1} \sum_{j=1}^{2^d} |R_j|^2,\end{aligned}$$

using (2.1). Now, by Jensen's inequality, $\sum_{j=1}^{2^d} |R_j|^2 \geq 2^{-d}(\sum_{j=1}^{2^d} |R_j|)^2 = 2^{-d}|R|^2$, and the claimed result follows with $\delta = 2^{-2d}\lambda_0/\lambda_1$.

Next we give bounds on expectations for $N_n^{\mathrm{ONG}}(k)$.

Lemma 6.2. *Let $d \in \mathbb{N}$. Suppose that* (2.1) *holds. Then there exist finite positive constants A, A', C, C' such that, for all $k \in \mathbb{N}$,*

$$A'\mathrm{e}^{-C'k} \leq \liminf_{n \to \infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \leq \limsup_{n \to \infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \leq A\mathrm{e}^{-Ck}. \tag{6.2}$$

Moreover, more precisely,

$$-\liminf_{k \to \infty} \left(k^{-1} \log \left(\liminf_{n \to \infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)]\right)\right) \leq 1, \tag{6.3}$$

and, in the case where f is the uniform density on S,

$$-\limsup_{k\to\infty}\left(k^{-1}\log\left(\limsup_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)]\right)\right) \geq \frac{1}{2}\log\left(1+(2^{2d}-1)^{-1}\right). \qquad (6.4)$$

Proof. First we prove the upper bound in (6.2), using an argument based in part on [1, §3.1]. Fix $i \in \mathbb{Z}_+$. Let $t_0 = i$ and for $j \in \mathbb{N}$ define recursively $t_j = \min\{t > t_{j-1} : X_t \in \mathcal{V}_{t-1}(i)\}$, so that $t_1, t_2, \ldots$ are the times at which edges to X_i are created. Following [1, p. 311], let $\mathcal{W}_j = \mathcal{V}_{t_j}(i)$.

Observe that if i has degree greater than k in the ONG on $(X_0, \ldots, X_n)$, $n \geq i$, then necessarily $t_k \leq n$, and so also $|\mathcal{V}_n(i)| \leq |\mathcal{V}_{t_k}(i)| = |\mathcal{W}_k|$, by (6.1). Hence,

$$\mathbb{P}[\deg_n(i) > k] \leq \mathbb{P}[|\mathcal{W}_k| \geq z] + \mathbb{P}[|\mathcal{V}_n(i)| \leq z], \text{ for any } z > 0. \qquad (6.5)$$

We bound each of the probabilities on the right-hand side of (6.5) in turn, and then optimize the choice of z.

By definition, X_{t_j} is distributed according to the density f, conditioned to fall in the convex set $\mathcal{V}_{t_j-1}(i) \subseteq \mathcal{V}_{t_{j-1}}(i) \subseteq S$. Hence Lemma 6.1 shows that $\mathbb{E}[|\mathcal{W}_j|] \leq (1-\delta)\mathbb{E}[|\mathcal{W}_{j-1}|]$, where $\delta \in (0,1)$ depends only on d and λ_0/λ_1, and

$$\mathbb{E}[|\mathcal{W}_j|] \leq (1-\delta)^j \mathbb{E}[|\mathcal{V}_i(i)|] = \frac{1}{i+1}(1-\delta)^j,$$

since the vector $(|\mathcal{V}_i(0)|, \ldots, |\mathcal{V}_i(i)|)$ is exchangeable and its components sum to 1, so $\mathbb{E}[|\mathcal{V}_i(j)|] = \frac{1}{i+1}$. Markov's inequality implies that, for any $z > 0$,

$$\mathbb{P}[|\mathcal{W}_j| \geq z] \leq \frac{1}{z}\frac{1}{i+1}(1-\delta)^j. \qquad (6.6)$$

The final term in (6.5) is bounded above by (5.2). Combining (6.5) with (6.6) and (5.2), we obtain, for any $z > 0$, $\mathbb{P}[\deg_n(i) > k] \leq \frac{1}{z}\frac{1}{i+1}(1-\delta)^k + Cnz$, where $C < \infty$ depends only on d, S, and λ_1. The optimal bound is obtained on taking $z = (1-\delta)^{k/2}/\sqrt{Cn(i+1)}$, and we conclude

$$\mathbb{P}[\deg_n(i) > k] \leq 2(1-\delta)^{k/2}\sqrt{\frac{Cn}{i+1}}. \qquad (6.7)$$

The upper bound in (6.2) follows from (6.7), since

$$\mathbb{E}[N_n^{\mathrm{ONG}}(k)] = \sum_{i=0}^{n} \mathbb{P}[\deg_n(i) \geq k] \leq C' n (1-\delta)^{k/2},$$

for some $C' < \infty$ not depending on k or n. The statement (6.4) also follows, since when $\lambda_0 = \lambda_1$, we have from the proof of Lemma 6.1 that we may take $\delta = 2^{-2d}$.

To prove the lower bound in (6.2) as well as (6.3), we use a similar idea to that briefly outlined for the analogous argument in [1, p. 311], but filling in the details takes some work, and we must be more careful with our estimates to obtain the quantitative bound (6.3). First note that, for $j > i$, the (unconditional) probability that X_j is joined to X_i is $\mathbb{P}[\eta_1(j) = i] = \mathbb{P}[X_j \in \mathcal{V}_{j-1}(i)] = 1/j$. Write $d_n(i) := \mathbb{E}[\deg_n(i)]$. Then, for $i \in \mathbb{N}$,

$$d_n(i) = 1 + \sum_{j=i+1}^{n} \mathbb{P}[\eta_1(j) = i] \geq \sum_{j=i}^{n} \frac{1}{j} \geq \int_i^n \frac{1}{y} \mathrm{d}y = \log(n/i).$$

Let $\theta > 1$. For $k \in \mathbb{Z}_+$, let $H_{n,k}^\theta := \mathbb{N} \cap [1, \mathrm{e}^{-\theta k} n]$. Then for any $i \in H_{n,k}^\theta$, $d_n(i) \geq \log(n/i) \geq \theta k$. It follows that

$$\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \geq \sum_{i \in H_{n,k}^\theta} \mathbb{P}[\deg_n(i) \geq k] \geq \sum_{i \in H_{n,k}^\theta} \mathbb{P}[\deg_n(i) \geq \theta^{-1} d_n(i)]. \tag{6.8}$$

Let $w \in (1, \infty)$, to be specified later. Then $w > 1 > 1/\theta$, and

$$\begin{aligned} w d_n(i) \mathbb{P}[\deg_n(i) \geq \theta^{-1} d_n(i)] &\geq \mathbb{E}[\deg_n(i) \mathbf{1}\{\deg_n(i) \geq \theta^{-1} d_n(i)\}] \\ &\quad - \mathbb{E}[\deg_n(i) \mathbf{1}\{\deg_n(i) > w d_n(i)\}] \\ &\geq \left(1 - \theta^{-1}\right) d_n(i) - \mathbb{E}[\deg_n(i) \mathbf{1}\{\deg_n(i) > w d_n(i)\}], \end{aligned} \tag{6.9}$$

using the fact that $\mathbb{E}[X \mathbf{1}\{X \geq x\}] \geq \mathbb{E}[X] - x$ for any $x \geq 0$ and any nonnegative random variable X. By the Cauchy–Schwarz inequality, the final term in (6.9) satisfies

$$\mathbb{E}[\deg_n(i) \mathbf{1}\{\deg_n(i) > w d_n(i)\}] \leq \left(\mathbb{E}[\deg_n(i)^2] \mathbb{P}[\deg_n(i) > w d_n(i)]\right)^{1/2}. \tag{6.10}$$

We claim that, given $\theta > 1$, there exists $w = w(\theta) \in (1, \infty)$ such that

$$\sup_{i \in H_{n,k}^\theta} \left(\mathbb{E}[\deg_n(i)^2] \mathbb{P}[\deg_n(i) > w d_n(i)]\right)^{1/2} \leq \mathrm{e}^{-\theta k}, \text{ for all } n, k \in \mathbb{N}. \tag{6.11}$$

Given (6.11), which we verify at the end of this proof, we obtain from (6.9), (6.10), and (6.11) that, for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$,

$$\begin{aligned} w \inf_{i \in H_{n,k}^\theta} \mathbb{P}[\deg_n(i) \geq \theta^{-1} d_n(i)] &\geq \left(1 - \theta^{-1}\right) - \mathrm{e}^{-\theta k} \sup_{i \in H_{n,k}^\theta} \frac{1}{d_n(i)} \\ &\geq \left(1 - \theta^{-1}\right) - \frac{\mathrm{e}^{-\theta k}}{\theta k}, \end{aligned} \tag{6.12}$$

using the fact that $d_n(i) \geq \theta k$ for $i \in H^\theta_{n,k}$. To prove the lower bound in (6.2), it is enough to fix $\theta = 2$. Then (6.12) becomes, for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$,

$$w \inf_{i \in H^2_{n,k}} \mathbb{P}[\deg_n(i) \geq \tfrac{1}{2} d_n(i)] \geq \frac{1}{2}\left(1 - \mathrm{e}^{-2}\right) \geq \frac{3}{8},$$

say, where $w = w(2)$ is constant. Hence from (6.8) we obtain, for all $n, k \in \mathbb{N}$,

$$\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \geq w^{-1} \sum_{i \in H^2_{n,k}} \frac{3}{8} \geq \frac{3}{8w}\left(\mathrm{e}^{-2k} n - 1\right),$$

which gives $\liminf_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \geq \frac{3}{8w}\mathrm{e}^{-2k}$.

To prove (6.3), we adapt the preceding argument. For any $\theta > 1$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, the final expression on the right-hand side of (6.12) exceeds $\frac{1-\theta^{-1}}{2} > 0$, say. Then, similarly to before, we obtain, for all $k \geq k_0$ and $n \in \mathbb{N}$,

$$\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \geq w^{-1} \sum_{i \in H^\theta_{n,k}} \left(\frac{1-\theta^{-1}}{2}\right) \geq w^{-1}\left(\frac{1-\theta^{-1}}{2}\right)\left(\mathrm{e}^{-\theta k} n - 1\right).$$

First letting $n \to \infty$ and then $k \to \infty$, it follows that

$$\limsup_{k\to\infty}\left(-k^{-1}\log\left(\liminf_{n\to\infty} n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)]\right)\right) \leq \theta.$$

Since $\theta > 1$ was arbitrary, (6.3) follows.

It remains to establish the claim (6.11). To this end, an application of (6.7) shows that, for constants $C_1, C_2 < \infty$ and $c > 0$, for all $n \in \mathbb{N}$ and $1 \leq i \leq n$,

$$\mathbb{E}[\deg_n(i)^2] = \sum_{k=0}^\infty \mathbb{P}[\deg_n(i) > \sqrt{k}] \leq C_1\sqrt{\frac{n}{i}}\sum_{k=0}^\infty \mathrm{e}^{-c\sqrt{k}} \leq C_2\sqrt{\frac{n}{i}}.$$

Another application of (6.7) shows that, for some constant $C_3 < \infty$, for any $w > 0$,

$$\mathbb{P}[\deg_n(i) > w d_n(i)] \leq C_3\sqrt{\frac{n}{i}}\mathrm{e}^{-cw\log(n/i)} = C_3\left(\frac{n}{i}\right)^{(1/2)-cw}.$$

Hence we obtain, for all $1 \leq i \leq n$,

$$\left(\mathbb{E}[\deg_n(i)^2]\mathbb{P}[\deg_n(i) > w d_n(i)]\right)^{1/2} \leq C_4\left(\frac{i}{n}\right)^{(cw-1)/2},$$

where $C_4 < \infty$ is constant. Taking $w > 3/c$, we have, for any $i \in H^\theta_{n,k}$,

$$C_4\left(\frac{i}{n}\right)^{(cw-1)/2} \leq C_4\mathrm{e}^{-\theta k}\mathrm{e}^{-(cw-3)\theta k/2},$$

since $i/n \leq \mathrm{e}^{-\theta k}$ for $i \in H^\theta_{n,k}$. In particular, for all $k \in \mathbb{N}$, we can choose w (depending on c, C_4 and θ) such that $C_4\mathrm{e}^{-(cw-3)\theta k/2} \leq C_4\mathrm{e}^{-(cw-3)\theta/2} \leq 1$. This verifies (6.11).

Next we have a concentration result for $N_n^{\mathrm{ONG}}(k)$.

Lemma 6.3. *Let $d \in \mathbb{N}$. Suppose that* (2.1) *holds. Then*

$$\limsup_{n\to\infty} n^{-1} \sup_{k\in\mathbb{N}} |N_n^{\mathrm{ONG}}(k) - \mathbb{E}[N_n^{\mathrm{ONG}}(k)]| = 0, \ \textit{a.s.} \tag{6.13}$$

Proof. We use the resampling idea described before Lemma 5.3 and a modification of the Azuma–Hoeffding inequality. Set $D_{n,i} = \mathbb{E}[N_n^{\mathrm{ONG}}(k) \mid \mathcal{G}_i] - \mathbb{E}[N_n^{\mathrm{ONG}}(k) \mid \mathcal{G}_{i-1}]$ where $\mathcal{G}_i = \sigma(X_0, X_1, \ldots, X_i)$. In words, $-D_{n,i}$ is equal to the expected (conditional on $\mathcal{G}_i$) change in $N_n^{\mathrm{ONG}}(k)$ on independently resampling X_i, and $D_{n,i}$, $1 \le i \le n$ is a martingale difference sequence with $\sum_{i=1}^n D_{n,i} = N_n^{\mathrm{ONG}}(k) - \mathbb{E}[N_n^{\mathrm{ONG}}(k)]$.

We bound $|D_{n,i}|$ in terms of $\deg_n(i)$ and $\deg_n^i(i)$, the degree of vertex i in the ONG on $\mathcal{X}_n$ and $\mathcal{X}_n^i$ respectively. On replacement of X_i by X_i', the degree of vertex i may change, leading to a change of ± 1 in $N_{n,i}^{\mathrm{ONG}}(k)$ compared to $N_n^{\mathrm{ONG}}(k)$. The degrees of at most $\deg_n(i) - 1$ other vertices increase (namely those vertices that gain incoming edges that were previously connected to X_i), while the degrees of at most $\deg_n^i(i) - 1$ vertices decrease (namely those vertices that lose incoming edges re-assigned to X_i').

Hence $|D_{n,i}| \le \deg_n(i) + \deg_n^i(i)$. Now, for any $r > 0$,

$$\mathbb{P}[|D_{n,i}| > r] \le \mathbb{P}[\deg_n(i) > r/2] + \mathbb{P}[\deg_n^i(i) > r/2] = 2\mathbb{P}[\deg_n(i) > r/2],$$

since $\deg_n^i(i)$ and $\deg_n(i)$ are identically distributed. Hence, by (6.7), $\mathbb{P}[|D_{n,i}| > D \log n] = O(n^{-5})$, uniformly in i, choosing $D \in (0, \infty)$ sufficiently large; note that this bound is also uniform in k. By a modification of the Azuma–Hoeffding inequality due to Chalker *et al.* [4, Lemma 1], it follows that

$$\mathbb{P}[|N_n^{\mathrm{ONG}}(k) - \mathbb{E}[N_n^{\mathrm{ONG}}(k)]| > r] \le \left(1 + \frac{4n}{r}\right) n^{-4} + 2\exp\left\{-\frac{r^2}{32D^2 n (\log n)^2}\right\},$$

for any $r > 0$. Taking $r = n^{3/4}$, say, shows that $\mathbb{P}[|N_n^{\mathrm{ONG}}(k) - \mathbb{E}[N_n^{\mathrm{ONG}}(k)]| > n^{3/4}] = O(n^{-3})$, uniformly in $k \in \{1, \ldots, n\}$, while for $k > n$, $N_n^{\mathrm{ONG}}(k) = 0$ a.s. Hence

$$\sum_{n=1}^{\infty} \sum_{k\in\mathbb{N}} \mathbb{P}[|N_n^{\mathrm{ONG}}(k) - \mathbb{E}[N_n^{\mathrm{ONG}}(k)]| > n^{3/4}] \le C \sum_{n=1}^{\infty} n^{-2} < \infty.$$

The Borel–Cantelli lemma now yields (6.13).

Proof of Theorem 2.1. Penrose [16, §3.4] showed that functionals such as counts of vertices of a given degree in the ONG satisfy *stabilization* (a form of local dependence).

This guarantees a law of large numbers of the form $n^{-1}N_n^{\mathrm{ONG}}(k) \to \mathbb{P}[\xi(0,U;\mathcal{H}) \geq k]$ *in probability*: concretely, one may apply results of [19] or [17]. The bounded convergence theorem gives $n^{-1}\mathbb{E}[N_n^{\mathrm{ONG}}(k)] \to \mathbb{P}[\xi(0,U;\mathcal{H}) \geq k]$, and Lemma 6.3 then shows that convergence in probability can be replaced by almost sure convergence. The L^1 convergence follows from the bounded convergence theorem again. Thus (2.3) holds.

Then, applying Lemma 6.2 with (2.3), (2.4) follows from (6.2). Given (2.3), the upper bound in (2.5) follows from (6.3). Similarly, the lower bound in (2.5) follows from (6.4), noting that the limit ρ_k is independent of the choice of f.

It is easy to see that ρ_k is nonincreasing with $\rho_1 = 1$. Since $\sum_{k\in\mathbb{N}} N_n^{\mathrm{ONG}}(k) = 2n$, twice the number of edges in the ONG, dividing both sides of this last equality by n and letting $n \to \infty$ we must have $\sum_{k\in\mathbb{N}} \rho_k = 2$; hence also $\lim_{k\to\infty} \rho_k = 0$. For the final statement of the theorem, we have from (6.7) that for any $k > 0$,

$$\mathbb{P}\Big[\max_{0\leq i\leq n} \deg_n(i) > k\Big] \leq (n+1) \max_{0\leq i\leq n} \mathbb{P}[\deg_n(i) > k] \leq Cn^{3/2}\mathrm{e}^{-ck},$$

for some absolute constants $c, C \in (0,\infty)$. Taking $k = D\log n$, we can choose $D \in (0,\infty)$ for which this last bound is $O(n^{-2})$; the Borel–Cantelli lemma then gives (2.6).

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